ON A CASE OF SMALL VIBRATIONS OF A PHYSICAL PENDULUM WITH A MOVING POINT OF SUPPORT

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The problem of small vibrations of a physical symmetric pendulum about the position of relative equilibrium is to be considered. The point of support moves close to the surface of the earth, and the parameters of the pendulum are so chosen that in the position of relative equilibrium its dynamic axis of symmetry coincides with the direction to the center of the earth.

Paper [1] establishes the following conditions to be satisfied by a heavy rigid body with an axis of dynamic symmetry and so fixed at one of the points along this axis (pendulum) that its axis of symmetry coincides with the direction to the center of the earth for arbitrary motions of the point of support on the surface of the earth.

1. The reduced length of the pendulum equals the radius of the earth:

$$\frac{A}{ma} = R_0 \tag{0.1}$$

Condition (0.1) is the well-known Schuler condition [2].

2. One of these two requirements is satisfied: either the projection of the absolute angular velocity of the pendulum on the direction to the center of the earth (in the position of relative equilibrium, coinciding with the axis of dynamic symmetry of the pendulum) at the instant of the beginning of motion $\omega_{z_0}(0) = 0$; or the moment of inertia of the pendulum with respect to its axis of dynamic symmetry $C = 0$.

If conditions 1 and 2 are satisfied, the axis of dynamic symmetry of the pendulum will always coincide with the direction to the center of the earth, provided the two were coincident at the instant of the beginning of motion.

This paper considers small vibrations of the axis of dynamic symmetry of the pendulum about the direction to the center of the earth, for the case when, at the instant of the beginning of motion, the axis of dynamic symmetry does not coincide with the direction to the center of the earth. The study of these vibrations permits us to judge the degree of stability of the position of relative equilibrium found in paper [1]. Initially, just as in [1], the point of support of the pendulum is assumed to move on the surface of the earth, which is taken to be a sphere, and the gravity of the earth is assumed to form a central field. Later (Section 4), the case of arbitrary motion of the point of support in the neighborhood of the surface of the earth is considered, and the earth's gravity field is taken to be non-central; in this case the compensating moments which must be applied to the pendulum (in addition to conditions 1 and 2) for a position of relative equilibrium to exist, are determined.

1. We introduce the following right-handed rectangular systems of coordinates: $O'\xi^*\eta^*\zeta^*$, the origin O' of which coincides (Fig. 1) with the point of support of the pendulum, and which moves arbitrarily on a fixed sphere S of radius R_0, which is concentric to the earth's surface; the orientation of the axis remains unchanged with respect to the fixed stars. The system of coordinates $O'\xi^*\eta^*\zeta^*$ thus moves in forward motion with respect to fixed stars.

The system $O'x_0y_0z_0$ is connected with the pendulum (Fig. 2) in its undisturbed motion in the position of relative equilibrium; the axis $O'z_0$ of this system of coordinates coincides with the axis of dynamic symmetry of the pendulum, and is directed from the center along the radius of the earth; the axis $O'x_0$ and $O'y_0$ lie in the tangential plane to the sphere S and therefore also to the earth. The system $O'x_0y_0z_0$ is thus Darboux's trihedron on the surface of the earth, connected with the point O'

The system $O'xyz$ is connected with the body of the pendulum (Fig. 2) in its disturbed motion about the position of relative equilibrium; the location of the system of coordinates $O'xyz$ with respect to the system $O'x_0y_0z_0$ is determined by two angles α and β (α and β are small) in accordance with the table of direction cosines*:

	x_0	y_0	z_0
x	1	0	$-\beta$
y	0	1	α
z	β	$-\alpha$	1

(1.1)

* The disturbed position of the pendulum is here determined by two angles, since we are interested in the motion (deviation from the direction to the center of the earth) of the axis of dynamic symmetry of the pendulum.

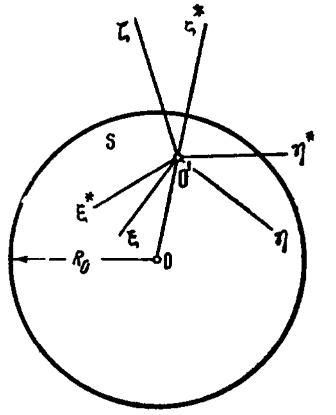

Fig. 1.

The axis $O'z$ is the axis of dynamic symmetry of the pendulum, and the center of gravity c of the pendulum has the coordinates

$$x_c = y_c = 0, \qquad z_c = -a \tag{1.2}$$

The axes of the system of coordinates $O'xyz$ are directed along the principal axes of the ellipsoid of inertia of the pendulum, which is an ellipsoid of rotation; therefore

$$I_{xy} = I_{yz} = I_{zx} = 0, \qquad I_{xx} = I_{yy} = A, \qquad I_{zz} = C \tag{1.3}$$

2. We form the equations of motion of the pendulum with respect to the system of coordinates $O'\xi^*\eta^*\zeta^*$ in projections on axes $O'xyz$; to this end we use Euler's equations, which in our case have the form:

$$\begin{aligned} A d\omega_x / dt + (C - A)\,\omega_y\omega_z &= M_x \\ A d\omega_y / dt - (C - A)\,\omega_x\omega_z &= M_y \\ C d\omega_z / dt &= M_z \end{aligned} \tag{2.1}$$

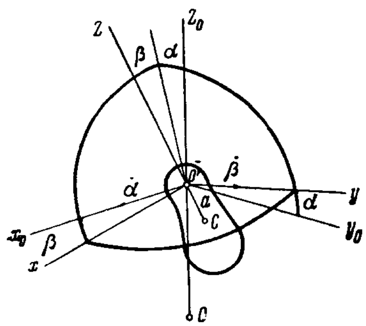

Fig. 2.

If ω_{x0}, ω_{y0}, ω_{z0} are the projections of the absolute angular velocity of the system of coordinates $O'x_0y_0z_0$ on its axes, then, taking (1.1) into consideration, we have the following expressions for the projections of the absolute angular velocity of the pendulum on the axes xyz fixed on it:

$$\begin{aligned}\omega_x &= \omega_{x_0} - \omega_{z_0}\beta + d\alpha / dt\\ \omega_y &= \omega_{y_0} + \omega_{z_0}\alpha + d\beta / dt\\ \omega_z &= \omega_{z_0} - \omega_{y_0}\alpha + \omega_{x_0}\beta\end{aligned} \qquad (2.2)$$

To calculate the moments M_x, M_y, M_z we note that in the present case they are composed of moments due to the gravity of the earth and to the inertia forces produced by the motion of the system of coordinates $O'\xi^*\eta^*\zeta^*$, in which the equations of motion of the pendulum are formulated. Assuming the attraction of the elementary masses of the pendulum due to earth, to be reduced to a single force **F** directed along the radius of the earth to its center* and having the magnitude $F = mg_0$, where m is the mass of the pendulum, and g_0 is the gravitational force of attraction by the earth on a unit mass placed on its surface, and also taking into consideration that the projections of the absolute velocity of Darboux's trihedron $O'x_0y_0z_0$ on its axes** are

$$v_{x_0} = R_0\omega_{y_0}, \qquad v_{y_0} = -R_0\omega_{x_0}, \qquad v_{z_0} = 0 \qquad (2.3)$$

and taking into account (1.2) and (1.3), we obtain,

$$\begin{aligned}M_x &= -mag_0\alpha + maR_0\left(\frac{d\omega_{x_0}}{dt} - \omega_{y_0}\omega_{z_0}\right) + maR_0(\omega_{x_0}^2 + \omega_{y_0}^2)\alpha\\ M_y &= -mag_0\beta + maR_0\left(\frac{d\omega_{y_0}}{dt} + \omega_{x_0}\omega_{z_0}\right) + maR_0(\omega_{x_0}^2 + \omega_{y_0}^2)\beta\\ M_z &= 0\end{aligned} \qquad (2.4)$$

Substituting the relationships (2.2) and (2.4) into (2.1), and omitting second-order terms, the equations of motion, after an appropriate collection of terms, may be written down as follows:

$$(A - maR_0)\left(\frac{d\omega_{x_0}}{dt} - \omega_{y_0}\omega_{z_0} + \omega_{y_0}^2\alpha\right) + C\omega_{y_0}(\omega_{z_0} - \omega_{y_0}\alpha + \omega_{x_0}\beta) +$$
$$+ C\omega_{z_0}\left(\frac{d\beta}{dt} + \omega_{z_0}\alpha\right) + A\frac{d^2\alpha}{dt^2} + (mag_0 - maR_0\omega_{x_0}^2 - A\omega_{z_0}^2)\alpha =$$
$$= A\left(\omega_{x_0}\omega_{y_0} + \frac{d\omega_{z_0}}{dt}\right)\beta + 2A\omega_{z_0}\frac{d\beta}{dt}$$

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* Special cases of the plane problem of small vibrations of a physical pendulum, taking into account the resulting moment of gravity forces, are considered in papers [3,4].

** The center of the earth is assumed to be a fixed point.

$$
\begin{aligned}
&(A - maR_0)\left(\frac{d\omega_{y_0}}{dt} + \omega_{x_0}\omega_{z_0} + \omega_{x_0}^2\beta\right) - C\omega_{x_0}(\omega_{z_0} - \omega_{y_0}\alpha + \omega_{x_0}\beta) - \\
&\quad - C\omega_{z_0}\left(\frac{d\alpha}{dt} - \omega_{z_0}\beta\right) + A\frac{d^2\beta}{dt^2} + (mag_0 - maR_0\omega_{y_0}^2 - A\omega_{z_0}^2)\beta = \\
&\qquad = A\left(\omega_{x_0}\omega_{y_0} - \frac{d\omega_{z_0}}{dt}\right)\alpha - 2A\omega_{z_0}\frac{d\alpha}{dt} \qquad (2.5) \\
&\qquad C\frac{d}{dt}(\omega_{z_0} - \omega_{y_0}\alpha + \omega_{x_0}\beta) = 0
\end{aligned}
$$

It follows from the third equation, (2.5), that

$$
C(\omega_{z_0} - \omega_{y_0}\alpha + \omega_{x_0}\beta) = C(\omega_{z_0} - \omega_{y_0}\alpha + \omega_{x_0}\beta)\,|_{t=0} = CH = \text{const}
$$

Taking this into account, the first two equations (2.5) take on the form:

$$
\begin{aligned}
&(A - maR_0)\left(\frac{d\omega_{x_0}}{dt} - \omega_{y_0}\omega_{z_0} + \omega_{y_0}^2\alpha\right) + CH\left(\omega_{y_0} + \frac{d\beta}{dt} + \omega_{z_0}\alpha\right) + \\
&+ A\frac{d^2\alpha}{dt^2} + (mag_0 - maR_0\omega_{x_0}^2 - A\omega_{z_0}^2)\alpha = A\left(\omega_{x_0}\omega_{y_0} + \frac{d\omega_{z_0}}{dt}\right)\beta + 2A\omega_{z_0}\frac{d\beta}{dt} \\
&(A - maR_0)\left(\frac{d\omega_{y_0}}{dt} + \omega_{x_0}\omega_{z_0} + \omega_{x_0}^2\beta\right) - CH\left(\omega_{x_0} + \frac{d\alpha}{dt} - \omega_{z_0}\beta\right) + \\
&+ A\frac{d^2\beta}{dt^2} + (mag_0 - maR_0\omega_{y_0}^2 - A\omega_{z_0}^2)\beta = A\left(\omega_{x_0}\omega_{y_0} - \frac{d\omega_{z_0}}{dt}\right)\alpha - 2A\omega_{z_0}\frac{d\alpha}{dt} \qquad (2.6)
\end{aligned}
$$

From (2.6) it immediately follows that the equations of motion have the trivial solution $\alpha = \beta \equiv 0$, that is the assumption regarding the relative equilibrium indicated above exists only when

$$
A = maR_0, \qquad CH = 0
$$

It is easily seen that the condition $H = 0$ is equivalent to the second condition 2°: $\omega_{z0} = 0$ when $t = 0$.

From (2.6), for $C = 0$ and $A = maR_0$ we obtain

$$
\begin{aligned}
\frac{d^2\alpha}{dt^2} + \omega_0^2\alpha &= (\omega_{x_0}^2 + \omega_{z_0}^2)\alpha + \left(\omega_{x_0}\omega_{y_0} + \frac{d\omega_{z_0}}{dt}\right)\beta + 2\omega_{z_0}\frac{d\beta}{dt} \\
\frac{d^2\beta}{dt^2} + \omega_0^2\beta &= (\omega_{y_0}^2 + \omega_{z_0}^2)\beta + \left(\omega_{x_0}\omega_{y_0} - \frac{d\omega_{z_0}}{dt}\right)\alpha - 2\omega_{z_0}\frac{d\alpha}{dt}
\end{aligned} \qquad (2.7)
$$

where $\omega_0^2 = g_0/R_0$.

For $H = 0$ and $A = maR_0$, omitting second-order quantities, we have

$$\frac{d^2\alpha}{dt^2} + \omega_0{}^2\alpha = \omega_{x_0}^2\alpha + \omega_{x_0}\omega_{y_0}\beta \qquad \left(\omega_0{}^2 = \frac{g_0}{R_0}\right) \qquad (2.8)$$
$$\frac{d^2\beta}{dt^2} + \omega_0{}^2\beta = \omega_{y_0}^2\beta + \omega_{x_0}\omega_{y_0}\alpha$$

Equations (2.7) and (2.8) describe the motion of the trihedron $Q'xyz$ with respect to the trihedron $O'x_0y_0z_0$, that is small vibrations of the pendulum, which satisfies conditions 1 and 2 in the vicinity of the position of relative equilibrium.

3. It is easily seen that equations (2.7) are invariant with respect to the transformation

$$\alpha' = \alpha\cos\varepsilon + \beta\sin\varepsilon$$
$$\beta' = -\alpha\sin\varepsilon + \beta\cos\varepsilon \qquad (3.1)$$

In passing from variables α and β to the variables α' and β', ω_{x0}, ω_{y0} and ω_{z0} must be correspondingly changed in the coefficients (2.8) to

$$\omega_{x_0}' = \omega_{x_0}\cos\varepsilon + \omega_{y_0}\sin\varepsilon, \quad \omega_{y_0}' = -\omega_{x_0}\sin\varepsilon + \omega_{y_0}\cos\varepsilon, \quad \omega_{z_0}' = \omega_{z_0} + \frac{d\varepsilon}{dt} \qquad (3.2)$$

If we put $d\epsilon/dt = -\omega_{z0}$, then equations (2.7) are transformed to equations (2.8); in this case, small vibrations are considered with respect to the trihedron $O'x_0'y_0'z_0$, which does not rotate about the axis $O'z_0$ (not rotating with respect to the azimuth, azimuthally free). Thus we may subsequently consider only the system of equations (2.8).

The study of a large class of small vibrations of equipment, which determine the vertical on a platform moving on the surface of the earth, can be reduced to equations (2.7) and (2.8); for example, if we put $\omega_{x0} = 0$ or $\omega_{y0} = 0$, then the equations of small vibrations of an ideal gyro-horizon compass are reducible to system (2.7) [5].

The study of the properties of solutions of equations (2.7) and (2.8), particularly of the divergence limits of the solution with respect to the initial conditions ("excitation" of pendulum), is thus of a certain practical interest.

The systems of equations (2.7) and (2.8) are, in general, systems with variable coefficients. Only for special choices of Darboux's trihedron $O'x_0'y_0'z_0$, and the law of motion of the point of support of the pendulum on the surface of the earth, may these equations be reduced to equations with constant coefficients.

For example, if the axis $O'x_0'$ is directed eastward along the parallel and the axis $O'y_0'$ northward along the meridian, and the motion of the point of support is along a constant latitude ϕ_0 with constant velocity v_0, then all coefficients in system (2.7) will be constant. In this case

$$\omega'_{x_0} = -\frac{d\varphi}{dt} = 0, \qquad \omega'_{y_0} = \left(u + \frac{v_0}{R_0}\right)\cos\varphi_0. \qquad \omega_{z_0} = \left(u + \frac{v_0}{R_0}\right)\sin\varphi_0$$

where u is the angular velocity of rotation of earth.

If the motion of the point of support is along a great circle of the sphere S with velocity v, then by placing the axes $O'x_0'$ and $O'z_0$ into the plane of this circle, from systems (2.7) or (2.8) we obtain

$$\frac{d^2\alpha'}{dt^2} + \omega_0^2\alpha' = 0, \qquad \frac{d^2\beta'}{dt^2} + \left(\omega_0^2 - \frac{v^2}{R_0^2}\right)\beta' = 0 \tag{3.3}$$

It is interesting to note that in this case the restoring moment with respect to the angle α' does not depend on the velocity of motion of the point of support.

For the case $v = v_0$ = const, the solution of (3.3) is given by harmonic oscillations with amplitudes to be determined from initial conditions. Otherwise, it is possible for parametric resonance to increase the amplitude of the angle β'.

To estimate the limits of possible divergence of solutions of the system of equations (2.7) or (2.8), with respect to initial conditions for finite time interval $[0, T]$ for the general case of motion of the point of support, reconstruct the solution of the system (2.8) by the method of successive approximations. We determine the relationship between the n and $(n-1)$ approximations as

$$\begin{aligned} \frac{d^2\alpha_{(n)}}{dt^2} + \omega_0^2\alpha_{(n)} &= \omega_{x_0}^2\alpha_{(n-1)} + \omega_{x_0}\omega_{y_0}\beta_{(n-1)} \\ \frac{d^2\beta_{(n)}}{dt^2} + \omega_0^2\beta_{(n)} &= \omega_{y_0}^2\beta_{(n-1)} + \omega_{x_0}\omega_{y_0}\alpha_{(n-1)} \end{aligned} \tag{3.4}$$

The zero approximation may be taken as a solution of equations*

$$\frac{d^2\alpha}{dt^2} + \omega_0^2\alpha = 0, \qquad \frac{d^2\beta}{dt^2} + \omega_0^2\beta = 0$$

Taking for simplicity

$$\alpha|_{=} = \beta|_{t=0} = a_0, \qquad \frac{d\alpha}{dt}\bigg|_{t=0} = \frac{d\beta}{dt}\bigg|_{t=0} = 0$$

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* In paper [5] this zero approximation was obtained by complex substitutions from a system which may be reduced to (2.8).

we obtain from (3.4)

$$\alpha_{(n)} = a_0 \cos \omega_0 t + \frac{1}{\omega_0} \int_0^t [\omega_{x_0}^2 \alpha_{(n-1)} + \omega_{x_0}\omega_{y_0}\beta_{(n-1)}] \sin \omega_0 (t-\tau)\, d\tau$$
$$\beta_{(n)} = a_0 \cos \omega_0 t + \frac{1}{\omega_0} \int_0^t [\omega_{y_0}^2 \beta_{(n-1)} + \omega_{x_0}\omega_{y_0}\alpha_{(n-1)}] \sin \omega_0 (t-\tau)\, d\tau \qquad (3.5)$$

The convergence and uniqueness of the solution determined by approximations (3.5) for bounded and continuous ω_{x0}, ω_{y0} are obvious [6].

It is easy to construct maximum estimates of the solution determined by approximations (3.5) for a time interval [0, T].

Let

$$\max \{\omega_{x_0}^2, \omega_{y_0}^2, |\omega_{x_0} \omega_{y_0}|\} < \mu$$

Then it follows from (3.5) that *

$$|\alpha_{(n)}| \leqslant a_0 \sum_{k=0}^{n'} \frac{1}{k!} \left(\frac{3\mu}{\pi\omega_0} T\right)^k$$

Hence

$$|\alpha| \leqslant a_0 \exp \frac{3\mu T}{\pi\omega_0} \qquad (3.6)$$

An analogous estimate is obtained for $|\beta|$.

The expression (3.6) may be used to estimate the divergence of a and β in time with respect to their initial values, if the form of the functions ω_{x0} and ω_{y0} is not known. In the opposite case, to sharpen the estimates, it is better to integrate the approximations (3.5) directly. For possible velocities v of the point of support on the surface of the earth

$$\frac{\mu}{\omega_0} \leqslant \frac{(v + uR_0)^2}{R_0 \sqrt{g_0 R_0}} \ll 1 \qquad (3.7)$$

and thus the approximations (3.5) converge rather rapidly. It is sufficient to use the second approximation for this estimate.

4. Let us consider the case of arbitrary motion of the point of support of the pendulum, close to the surface of the earth, and let us take into

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* Strictly speaking, the estimate is valid for $\pi(l + 1/2)/\omega_0 > T > \pi l/\omega_0$ where l is an integer; for $T \gg 2\pi/\omega_0$ this condition may be neglected. It may also be omitted if the interval of integration [0, T] is broken up into portions equal to $\pi/2\omega_0$.

account that the gravity field of the earth is not central. (The vicinity of motion to the surface of the earth has to be understood in the sense that the acceleration of the earth's gravity may be assumed to be constant in magnitude. In this case ω_x, ω_y, ω_z, as before, are determined by formulas (2.2) in equation (2.1), while for the velocities and the moments we have

$$v_{x_0} = R\omega_{y_0}, \qquad v_{y_0} = -R\omega_{x_0}, \qquad v_{z_0} = \frac{dR}{dt} \tag{4.1}$$

$$M_x = -mag_0'\alpha + ma\left[R\left(\frac{d\omega_{x_0}}{dt} - \omega_{y_0}\omega_{z_0}\right) - 2\frac{dR}{dt}\,\omega_{x_0}\right] + \\ + maR\,(\omega_{x_0}^2 + \omega_{y_0}^2)\,\alpha + M_x^{(1)} + M_x^{(2)}$$

$$M_y = -mag_0'\beta + ma\left[R\left(\frac{d\omega_{y_0}}{dt} + \omega_{x_0}\omega_{z_0}\right) + 2\frac{dR}{dt}\,\omega_{y_0}\right] + \\ + maR\,(\omega_{x_0}^2 + \omega_{y_0}^2)\,\beta + M_y^{(1)} + M_y^{(2)} \tag{4.2}$$

where $M_x^{(1)}$, $M_y^{(1)}$ denote moments due to the horizontal component of the force of gravity, while $M_x^{(2)}$, $M_y^{(2)}$ denote additional compensating moments applied to the pendulum. g_0' is the radial acceleration of earth's gravity at a distance R from its center.

To determine the moments $M_x^{(1)}$, $M_y^{(1)}$, in addition to the systems of coordinates already considered we introduce Darboux's trihedron $O'\xi\eta\zeta$ is directed east along the parallel and the axis $O'\eta$ northward along the meridian. Then, assuming the earth to be an ellipsoid of revolution [7], we may write

$$M_x^{(1)} = -ab\sin 2\varphi\cos(\eta y_0) \\ M_y^{(1)} = -ab\sin 2\varphi\cos(\eta x_0) \tag{4.3}$$

Here ϕ is the geographic latitude of the point O'·

$$b = \frac{e^2 g_0 - u^2 d}{2}$$

where d and e are respectively the minor semi-axis and the Krasovskii eccentricity of earth's ellipsoid.

For

$$C = 0, \qquad A = maR \qquad (A = \text{const}) \tag{4.4}$$

From (2.1), (2.2), (4.2) and (4.3) we obtain the equations of small oscillations in the case considered:

$$\frac{d^2\alpha}{dt^2} + \left(\frac{g_0'}{R} + \frac{1}{R}\frac{d^2R}{dt^2}\right)\alpha = (\omega_{x_0}^2 + \omega_{z_0}^2)\,\alpha +$$
$$+\left(\omega_{x_0}\omega_{y_0} + \frac{d\omega_{z_0}}{dt}\right)\beta + 2\omega_{z_0}\frac{d\beta}{dt} + \frac{2}{R}\frac{dR}{dt}\omega_{x_0} - \frac{b}{R}\sin 2\varphi\cos(\eta y_0) + \frac{M_x^{(2)}}{maR}$$
$$\frac{d^2\beta}{dt^2} + \left(\frac{g_0'}{R} + \frac{1}{R}\frac{d^2R}{dt^2}\right)\beta = (\omega_{y_0}^2 + \omega_{z_0}^2)\,\beta + \qquad (4.5)$$
$$+\left(\omega_{x_0}\omega_{y_0} - \frac{d\omega_{z_0}}{dt}\right)\alpha - 2\omega_{z_0}\frac{d\alpha}{dt} - \frac{2}{R}\frac{dR}{dt}\omega_{y_0} - \frac{b}{R}\sin 2\varphi\cos(\eta x_0) + \frac{M_y^{(2)}}{maR}$$

Equations (4.5) have trivial solutions if the compensating moments $M_x^{(2)}$, $M_y^{(2)}$ are formed in accordance with the equations

$$M_x^{(2)} = ma\left(-2\frac{dR}{dt}\omega_{x_0} + b\sin 2\varphi\cos(\eta y_0)\right)$$
$$M_y^{(2)} = ma\left(2\frac{dR}{dt}\omega_{y_0} + b\sin 2\varphi\cos(\eta x_0)\right) \qquad (4.6)$$

that is, the axis of dynamic symmetry of the pendulum, if the earth's gravitational field is considered non-central and if the vertical components of the velocity of the platform are taken into account, will in that case be in a position of relative equilibrium coincident with the direction to the center of the earth only if the moments $M_x^{(2)}$ and $M_y^{(2)}$ are applied to the pendulum along the axes $O'x$ and $O'y$, which are determined by equations (4.6).

If the corrective moments are applied externally to the pendulum, then the equations of small oscillations will have the form:

$$\frac{d^2\alpha}{dt^2} + \left(\frac{g_0'}{R} + \frac{1}{R}\frac{d^2R}{dt^2}\right)\alpha = (\omega_{x_0}^2 + \omega_{z_0}^2)\,\alpha + \left(\omega_{x_0}\omega_{y_0} + \frac{d\omega_z}{dt}\right)\beta + 2\omega_{z_0}\frac{d\beta}{dt} + \Delta_\alpha$$
$$\frac{d^2\beta}{dt^2} + \left(\frac{g_0'}{R} + \frac{1}{R}\frac{d^2R}{dt^2}\right)\beta = (\omega_{y_0}^2 + \omega_{z_0}^2)\,\beta + \left(\omega_{x_0}\omega_{y_0} - \frac{d\omega_{z_0}}{dt}\right)\alpha - 2\omega_{z_0}\frac{d\alpha}{dt} + \Delta_\beta \qquad (4.7)$$

where Δ_α and Δ_β are the non-compensating remainders of the disturbing moments.

Of particular interest is the case when the compensating moments $M_x^{(2)}$, $M_y^{(2)}$ are formed in making use of the coordinates of the instantaneous location of the object, which may be given by the pendulum itself.

In this case, also taking into account a possible error in satisfying the second condition (4.4) in the amount ΔR and a possible error $\Delta dR/dt$ in dR/dt for the formation of compensating moments, and also omitting small changes in the corrections due to the gravitational field of the earth being non-central, by introducing the change of variables

(3.1) and putting $d\epsilon/dt = -\omega_{z0}$ for small oscillations we obtain the following equations:

$$\frac{d^2}{dt^2}(R\alpha') + \frac{g_0'}{R}(R\alpha') = \omega_{x_0}'^2(R\alpha') + \omega_{x_0}'\omega_{y_0}'(R\beta') - \Delta R\frac{d\omega_{x_0}'}{dt} - 2\omega_{x_0}'\Delta\frac{dR}{dt} \quad (4.8)$$

$$\frac{d^2}{dt^2}(R\beta') + \frac{g_0'}{R}(R\beta') = \omega_{y_0}'^2(R\beta') + \omega_{x_0}'\omega_{y_0}'(R\alpha') - \Delta R\frac{d\omega_{y_0}'}{dt} - 2\omega_{y_0}'\Delta\frac{dR}{dt}$$

where ω_{x0}' ω_{y0}' ω_{z0}' are determined by formulas (3.2).

In studying system (4.8) we may assume $g_0'/R = g_0/R_0$ = const. To estimate the possible divergence of the solutions of the homogeneous system (4.8) with respect to initial conditions, we may then use relationships (3.5) or (3.6).

Owing to the fact that for the usual velocities of the point of support of the pendulum this divergence is small, in accordance with (3.6) and (3.7), to estimate the influence of right-hand sides in equations (4.8) we may consider the equations

$$\frac{d^2}{dt^2}(R\alpha') + \frac{g_0}{R_0}(R\alpha') = -\Delta R\frac{d\omega_{x_0}'}{dt} - 2\omega_{x_0}'\Delta\frac{dR}{dt}$$

$$\frac{d^2}{dt^2}(R\beta') + \frac{g_0}{R_0}(R\beta') = -\Delta R\frac{d\omega_{y_0}'}{dt} - 2\omega_{y_0}'\Delta\frac{dR}{dt} \quad (4.9)$$

In conclusion we remark that if the pendulum is not subjected to moments which compensate the disturbing effect of the horizontal component of the earth's gravity, then, since the change of this component in time $2\pi/\omega_0$ is small, the pendulum, for usual velocities of motion of the point of support, will be determined with great accuracy as being along the force of attraction of the earth.

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